Section 5.5: The Substitution Rule.

Objective: In this lesson, you learn

- ☐ how to replace a relatively complicated integral by a simpler integral using the Substitution Rule;
- ☐ how to replace a relatively complicated definite integral by a simpler definite integral using the Substitution Rule.

I. The Substitution Rule

In order to evaluate certain types of integrals, we introduce the **Substitution Rule**. But first, recall that if $u = f(x)$, then the differential is $du = f'(x)\, dx$. Also,

$\frac{du}{dx} = f'(x) \to du = f'(x)\, dx$

Recall: The chain Rule

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable, then

$$\frac{d}{dx}\left((f \circ g)(x)\right) = \frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x).$$

In terms of differential, if $y = f(g(x))$ then

$$dy = f'(g(x))\, g'(x)\, dx$$

Problem: Find

$$\int -2x e^{-x^2} dx$$

What if we think of the "dx" as a differential? If $u = e^{-x^2}$ what is the differential du?

$$u = e^{-x^2} \longrightarrow du = \boxed{-2x e^{-x^2}\, dx}$$

$f(x) = e^{g(x)}$

$f'(x) = g'(x)\, e^{g(x)}$

1. let $u = e^{-x^2}$, then $du = -2x e^{-x^2} dx$. so

$$\int -2x e^{-x^2} dx = \int du = u + C = e^{-x^2} + C$$

2. let $u = -x^2$, then $du = -2x\, dx$. so

$$\int -2x e^{-x^2} dx = \int e^{-x^2}(-2x\, dx) = \int e^{u}\, du = e^{u} + C = e^{-x^2} + C$$

The Substitution Rule (AKA undoing the Chain Rule)

This method of integrating works whenever we have an integral that we can write in the form

$$\int f\left(g\left(x\right)\right) g'\left(x\right) dx.$$

The Substitution Rule: If $u = g\left(x\right)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\left(g\left(x\right)\right) g'\left(x\right) dx = \int f\left(u\right) du.$$

Note that if $u = g\left(x\right)$, then $du = g'\left(x\right) dx$, so a way to remember the Substitution Rule is to think that dx and du are differentials.

Note:

1. This rule is a reversal of the chain rule.
2. The substitution rule says that we can work with "dx" and "du" that appear after the $\int$ symbols if they were differential.
3. The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral.
4. The main challenge in using the Substitution Rule is to think of an appropriate substitution. So you should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor).
5. Then check the answer by differentiating to obtain the original integrand.

Example 1: Evaluate the following

a. $\int 3x^2 \left(x^3+1\right)^4 dx$, $\qquad u = x^3 + 1$

u

x

PR.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

let $u = x^3 + 1 \longrightarrow du = 3x^2\, dx$

$$= \int \underbrace{3x^2 \left(\underbrace{x^3+1}_{u}\right)^4 dx}_{du} = \int u^4\, du$$

$$= \frac{u^{4+1}}{4+1} + C$$

$$= \frac{u^5}{5} + C = \boxed{\frac{(x^3+1)^5}{5} + C.}$$

$\boxed{u = x^3+1} \longrightarrow du = 3x^2 dx \Rightarrow \boxed{dx = \frac{du}{3x^2}}$

$$\int 3x^2 (x^3+1)^4 dx = \int \cancel{3x^2} \cdot (u)^4 \cdot \frac{du}{\cancel{3x^2}} = \int u^4\, du$$

$\frac{1}{x}$

b. $\int \frac{1}{ax+b}\,dx$

$\boxed{u = ax+b} \rightarrow du = a\,dx \rightarrow \boxed{dx = \frac{du}{a}}$

$\int c f(x)\,dx$
$c\int f(x)\,dx$

$$\int \frac{1}{ax+b}\,dx = \int \frac{1}{u}\,\frac{du}{a}$$
$$= \frac{1}{a}\int \frac{1}{u}\,du$$
$$= \frac{1}{a}\ln|u| + C$$
$$= \boxed{\frac{1}{a}\ln|ax+b| + C}$$

$\int \frac{1}{x}\,dx$
$\ln|x| + C$

c. $\int \frac{\cos(\pi/x)}{x^2}\,dx$

$\boxed{u = \frac{\pi}{x}} \rightarrow u = \pi x^{-1} \Rightarrow du = -\pi x^{-2}\,dx$

$\frac{x^2}{-\pi}\,du = \frac{-\pi}{x^2}\,dx\;\frac{x^2}{-\pi}$

$\boxed{dx = \frac{x^2}{-\pi}\,du}$

$\frac{1}{x} = x^{-1}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1})$
$= -x^{-2}$
$= \frac{-1}{x^2}$

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \frac{\cos(u)}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{-\pi}\cdot du$$
$$= \frac{-1}{\pi}\int \cos u\,du$$
$$= \frac{-1}{\pi}\sin u + C$$
$$= \boxed{\frac{-1}{\pi}\sin\left(\frac{\pi}{x}\right) + C}$$

d. $\int \sec^2 2\theta \, d\theta$

$\boxed{u = 2\theta} \to du = 2 \cdot d\theta$

$\boxed{\frac{du}{2} = d\theta}$

$\frac{d}{dx} \tan x = \sec^2 x + C$

$\sec^2 2\theta$

$\cancel{= 2 \sec^2 \theta}$

$\sin 90^\circ = 1$

$2 \sin 45^\circ = 2 \cdot \frac{1}{\sqrt{2}}$

$$\int \sec^2 2\theta \, d\theta = \int \sec^2 u \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int \sec^2 u \, du$$

$$= \frac{1}{2} \tan u + C = \boxed{\frac{1}{2} \tan 2\theta + C}$$

e. $\int \tan t \, dt$

$$= \int \frac{\sin t}{\cos t} \, dt$$

$\frac{d}{dx} \cos x = -\sin x$

$u = \cos t \to du = -\sin t \, dt$

$\boxed{-du = \sin t \, dt}$

$$= \int \frac{1}{u} (-du) = -\int \frac{1}{u} \, du$$

$\ln A^n = n \ln A$

$$= -\ln |u| + C$$

$$= \boxed{-\ln |\cos t| + C}$$

$$= \ln |\cos t|^{-1} + C$$

$$= \ln \left| \frac{1}{\cos t} \right| + C$$

$$= \boxed{\ln |\sec t| + C}$$

f. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

$u = \ln x \longrightarrow du = \frac{1}{x}\,dx$

$$\int \frac{dx}{x\sqrt{\ln x}} = 2\int \frac{1}{2\sqrt{u}}\,du$$

$$= 2\cdot\sqrt{u} + C$$

$$= 2\sqrt{\ln x} + C$$

OR

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du$$

$$= \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{u^{1/2}}{1/2} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{\ln x} + C$$

g. $\displaystyle\int x\sqrt{x^2+20}\,dx$

$u = x^2+20 \longrightarrow du = 2x\,dx \Rightarrow \boxed{\frac{du}{2} = x\,dx}$

$$= \int x\sqrt{x^2+20}\,dx = \int \sqrt{u}\;\frac{du}{2}$$

$$= \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{\cancel{2}}\,\frac{u^{3/2}}{3/\cancel{2}} + C = \boxed{\frac{1}{3}\left(x^2+20\right)^{3/2} + C}$$

h. $\int x^3\sqrt{x^2+20}\,dx$

$$\boxed{u = x^2+20} \longrightarrow du = 2x\,dx \longrightarrow \boxed{dx = \frac{du}{2x}}$$

$$\boxed{x^2 = u-20}$$

$$\int x^3\sqrt{x^2+20}\,dx = \int x^{\cancel{3}\,2}\cdot\sqrt{u}\cdot\frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int x^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int (u-20)\,u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - 20\,u^{1/2}\,du$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{5}u^{5/2} - 20\frac{\cancel{2}}{3}u^{3/2}\right) + C$$

$$= \frac{1}{5}(x^2+20)^{5/2} - \frac{20}{3}\cdot(x^2+20)^{3/2} + C.$$

i. $\int \frac{z^2}{\sqrt{1-z}}\,dz$

$$u = 1-z \longrightarrow du = -dz \longrightarrow \boxed{dz = -du}$$

$$\boxed{z = 1-u}$$

$$\int \frac{z^2}{\sqrt{1-z}}\,dz = \int \frac{z^2}{\sqrt{u}}\cdot(-du)$$

$$= -\int \frac{z^2}{\sqrt{u}}\,du$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$= -\int (1-u)^2\cdot u^{-1/2}\,du$$

$$= -\int (1 - 2u + u^2)\,u^{-\frac{1}{2}}\,du$$

$$= -\int u^{-\frac{1}{2}} - 2u^{1/2} + u^{3/2}\,du$$

$$= -\left(2u^{1/2} - 2\cdot\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) + C$$

$$= -\left(2\sqrt{1-z} - \frac{4}{3}(1-z)^{3/2} + \frac{2}{5}(1-z)^{5/2}\right) + C$$

$$\int \sec x \, dx$$

$$= \int \sec x \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

II. Substitution Rlue for definite integrals:

When evaluating a definite integral by substitution, two methods are possible:

a. Evaluate the integral first and then use the Fundamental Theorem.

b. Change the limits of integration when the variable is changed.

The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

i.e. make the substitution and change the limits of the integration at the same time.

Remark: When we make substitution $u = g(x)$, then the interval $[a, b]$ on the $x-$axis becomes the interval $[g(a), g(b)]$ on the $u-$axis.

Example 2: Evaluate the following

a. $\displaystyle\int_e^{e^2} \frac{(\ln x)^2}{x}\, dx$

1. $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$ $\quad u = \ln x \longrightarrow du = \frac{1}{x}\, dx$

$$= \int u^2\, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

$$\int_e^{e^2} \frac{(\ln x)^2}{x}\, dx = \frac{(\ln x)^3}{3}\Big|_e^{e^2} = \frac{(\cancel{\ln e^2})^3}{3} - \frac{(\cancel{\ln e})^3}{3}$$

$$= \frac{2^3}{3} - \frac{1}{3} = 7/3$$

OR

2. $\displaystyle\int_{x=e}^{x=e^2} \frac{(\ln x)^2}{x}\, dx,$ $\quad u = \ln x \longrightarrow du = \frac{1}{x}\, dx$

$x = e \to u = \ln e = 1$

$x = e^2 \to u = \cancel{\ln e^2} = 2$

$$= \int_{u=1}^{u=2} u^2\, du = \frac{u^3}{3}\Big|_{u=1}^{u=2} = \frac{2^3}{3} - \frac{1}{3} = 7/3$$

b. $\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

set $u = e^z + z \rightarrow du = e^z + 1 \; dz$

1. $\int \frac{e^z+1}{e^z+z} dz = \int \frac{du}{u} = \ln|u| + c = \ln|e^z+z| + c$

$\int_0^1 \frac{e^z+1}{e^z+z} dz = \ln|e^z+z| \Big|_0^1 = \boxed{\ln|e+1|} - \cancel{\ln|1|}$ 0

OR

2. $z=0 \rightarrow u = e^0 + 0 = 1$,

$z=1 \rightarrow u = e^1+1 = e+1$

$\int_{z=0}^{z=1} \frac{e^z+1}{e^z+z} dz = \int_{u=1}^{u=e+1} \frac{du}{u} = \ln|u| \Big|_{u=1}^{u=e+1}$ 0

$= \ln|e+1| - \cancel{\ln|1|}$

c. $\int_\pi^{2\pi} \cos 2t \, dt$

$u = 2t \longrightarrow du = 2\,dt \rightarrow dt = \frac{du}{2}$

$t = \pi \longrightarrow u = 2\cdot\pi = 2\pi$

$t = 2\pi \longrightarrow u = 2\cdot 2\pi = 4\pi$

$\int_{t=\pi}^{t=2\pi} \cos 2t \, dt = \int_{u=2\pi}^{u=4\pi} \cos u \frac{du}{2} = \frac{1}{2} \sin u \Big|_{u=2\pi}^{u=4\pi}$

$= \frac{1}{2}\left(\sin(4\pi) - \sin(2\pi)\right)$

$= 0$

III. Symmetry:

The next theorem uses the Substitution Rule for Definite Integrals to simplify the calculation of functions that possess symmetry properties.

Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

a. If f is even [that is, $f(-x) = f(x)$], then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

b. If f is odd [that is, $f(-x) = -f(x)$], then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

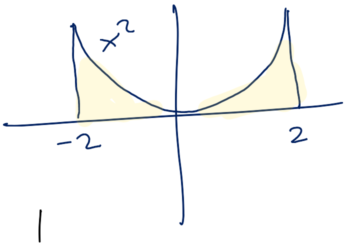
cos x is even

sin x is odd

$f(x) = x^2$
$f(-1) = (-1)^2 = 1$
$f(1) = (1)^2 = 1$

$f(x) = x^3$
$f(-1) = (-1)^3 = -1$
$f(1) = (1)^3 = 1$
$f(-1) = -f(1)$

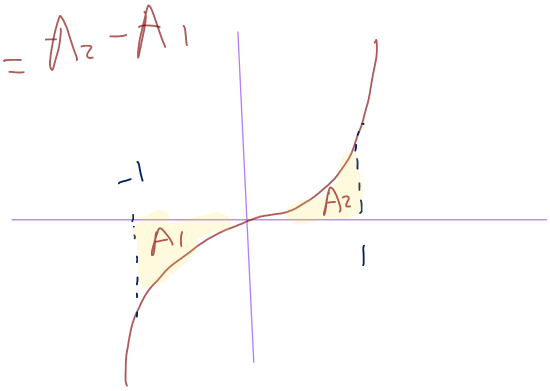
a.



$$\int_{-2}^{2} x^2\, dx = 2\int_{0}^{2} x^2\, dx$$

b.

$$\int_{-1}^{1} x^3\, dx = 0 = A_2 - A_1$$



Example 3: Evaluate the following

a. $\int_{-3}^{3} (3x^2 + 4)\, dx$

$f(-\cancel{x}) = 3(-x)^2 + 4$
$= 3x^2 + 4 = f(x)$
f is an even funct.

$= 2 \cdot \int_0^3 3x^2 + 4 \, dx$

$= 2 \cdot \left(x^3 + 4x \Big|_0^3 \right)$

$= 2\left[(3^3 + 4 \cdot 3) - (0) \right]$

b. $\int_{-e}^{e} \frac{e^{-u^2} \sin u}{u^2 + 10}\, du$

$\sin(-x) = -\sin x$

$f(-u) = \frac{e^{-(-u)^2} \sin(-u)}{(-u)^2 + 10}$

$= \frac{e^{-u^2}(-\sin(u))}{u^2 + 10}$

$= - \frac{e^{-u^2} \sin(u)}{u^2 + 10} = -f(u)$

f is odd $\Rightarrow$ $\int_{-e}^{e} \frac{e^{-u^2} \sin u}{u^2 + 10}\, du = 0$